विध्न विचारत भीरु जन, नहीं आरम्भे काम, विपति देख छोड़े तुरंत मध्यम मन कर श्याम।
पुरुष सिंह संकल्प कर, सहते विपति अनेक, 'बना' न छोड़े ध्येय को, रघुबर राखे टेक।।

रचितः मानव धर्म प्रणेता
सद्‌गुरु श्री रणछोड़दासजी महाराज

STUDY PACKAGE

Subject : Mathematics

Topic : HIGHLIGHTS ON PARABOLA, ELLIPSE & HYPERBOLA

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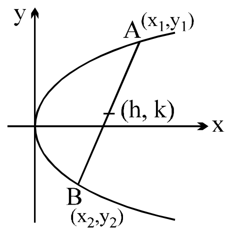
HIGHLIGHTS ON PARABOLA, ELLIPSE & HYPERBOLA

1.HIGHLIGHTS OF PARABOLA

CHORD WITH A GIVEN MIDDLE POINT : General method for finding the equation of a chord of any conic with middle point (h, k).



Example: Let the parabola be $y^2 = 8x$ and $(h, k) \equiv (2, -3)$
we have to find the equation of AB $\quad y + 3 = m(x - 2) \quad$ ....(1)

now $\quad y_1^2 = 4ax_1$

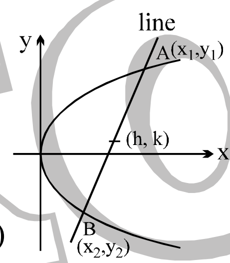
and $\quad y_2^2 = 4ax_2$

———————Subtraction

$$y_1^2 - y_2^2 = 4a(x_1 - x_2) \qquad \text{or} \qquad \frac{y_1 - y_2}{x_1 - x_2} = \frac{4a}{y_1 + y_2} = m$$

but $\quad y_1 + y_2 = 2k = -6$ and $4a = 8 \quad \therefore \quad m = \frac{8}{-6} = -\frac{4}{3}$

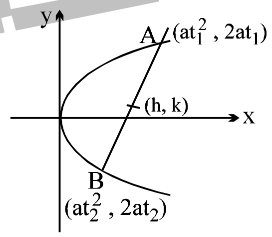
Hence equation of AB = $y + 3 = -\frac{4}{3}(x - 2) \Rightarrow \quad 4x + 3y + 1 = 0 \quad$]



Conversely : To find the mid point of given chord :Let the equation of the line be $4x + 3y + 1 = 0$ given.To find the mid point (h, k) of AB

here $\quad m = -\frac{4}{3} = \frac{4a}{y_1 + y_2} \Rightarrow \quad -\frac{4}{3} = \frac{8}{2k} \quad \Rightarrow \quad k = -3$

since $4h + 3k + 1 = 0 \Rightarrow 4h - 9 + 1 = 0 \quad \Rightarrow \quad h = 2$ hence M is $(2, -3)$



For a parabola in particular

Equation of AB $\qquad y - k = m(x - h) \qquad$ ....(1)

But $\quad m_{AB} = \frac{2}{t_1 + t_2} \qquad$ also $\quad 2k = 2a(t_1 + t_2)$

$(t_1 + t_2) = \frac{k}{a} \qquad \therefore \qquad m_{AB} = \frac{2a}{k} \quad$ Hence equation of a chord whose mid point is (h, k)

$y - k = \frac{2a}{k}(x - h) \qquad$ ....(2) or $\qquad 4ah + ky - k^2 = 2a(x - h) + 4ah = 2a(x + h)$

$$\underbrace{ky - 2a(x + h)}_{T} = \underbrace{k^2 - 4ah}_{S_1}$$

EXAMPLES : Ex-1Find the locus of the middle point of chords of the parabola $y^2 = 4ax$ which

(i) passes through the focus $\qquad$ [Ans. $y^2 = 2a(x - a)$]

(ii) are normal to the parabola $\qquad$ [Ans. $y^2(y^2 - 2ax + 4a^2) + 8a^4 = 0$]

(iii)subtend a constant angle α at the vertex(Homogenise)Ans.$(8a^2+y^2-2ax)^2\tan^2\alpha = 16a^2(4ax - y^2)$]

(iv)are of given length (say $2l$) $\qquad$ (v) are such that the normals at their extremities meet on the parabola $\qquad$ [Ans. $y^2 = 2a(x + 2a)$; Hint : use $t_1t_2 = 2$]

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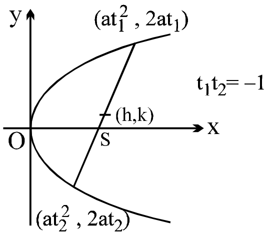
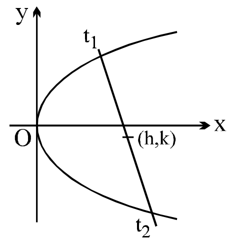


[Sol. (i) $2h = a(t_1^2 + t_2^2)$ ....(1)

$2k = 2a(t_1 + t_2)$ .....(2) $k^2 = a^2[t_1^2 + t_2^2 + 2t_1t_2]$

$k^2 = a^2\left[\dfrac{2h}{a} - 2\right] = 2ah - 2a^2 \quad \therefore \quad y^2 = 2a(x - a)$

This could also be spelled as locus of the middle point of all focal chords of all the particles $y^2 = 4ax$. **or** Locus of the middle point of all the chord of contact of the pair of tangents drawn from any point on this directrix.



(ii) $2h = a(t_1^2 + t_2^2)$ ....(1) ; $k = a(t_1 + t_2)$ .....(2)

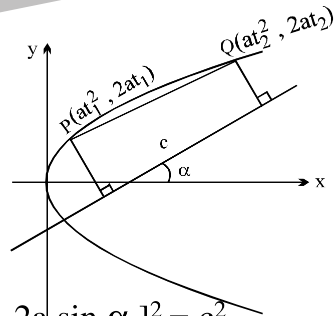
also $t_2 = -t_1 - \dfrac{2}{t_1} \Rightarrow t_1 + t_2 = -\dfrac{2}{t_1}$ using (2), $\dfrac{k}{a} = -\dfrac{2}{t_1} \Rightarrow t_1 = -\dfrac{2a}{k}$

$\therefore t_2 = +\dfrac{2a}{k} + \dfrac{2k}{2a} \Rightarrow t_2 = \dfrac{2a}{k} + \dfrac{k}{a} \therefore t_1t_2 = -\dfrac{2a}{k}\left[\dfrac{2a}{k} + \dfrac{k}{a}\right] = -2 - \dfrac{4a^2}{k^2}$ ....(3)

from (1) $2h = a[(t_1 + t_2)^2 - 2t_1t_2] = a\left[\dfrac{k^2}{a^2} + 2\left(2 + \dfrac{4a^2}{k^2}\right)\right] = a\left[\dfrac{k^2}{a^2} + 4 + \dfrac{8a^2}{k^2}\right] = a\left[\dfrac{k^4 + 4a^2k^2 + 8a^4}{a^2k^2}\right]$

$2ahk^2 = k^4 + 4a^2k^2 + 8a^4 \qquad k^2(k^2 - 2ah + 4a^2) + 8a^4 = 0$ Ans.]

Ex-2 A series of chords is drawn so that their projections on the straight line which is inclined at an angle α to the axis are of constant length c. Prove that the locus of their middle point is the curve $(y^2 - 4ax)(y\cos\alpha + 2a\sin\alpha)^2 + a^2c^2 = 0$



[Sol. Let $\hat{n} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$

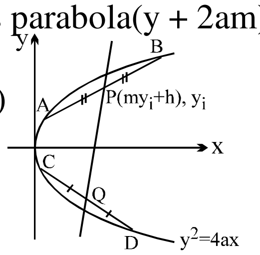
$\overrightarrow{PQ} = \vec{v} = (at_2^2 - at_1^2)\hat{i} + 2a(t_2 - t_2)\hat{j}$;;;$2h = a(t_1^2 + t_2^2)$; $a(t_1 + t_2) = k$

also projection of $\vec{v}$ on $\hat{n}$ = c $\left|\dfrac{\vec{v}\cdot\hat{n}}{|\hat{n}|}\right| = c$;;;

$\left|a(t_2^2 - t_1^2)\cos\alpha + 2a(t_2 - t_2)\sin\alpha\right| = c$;;;$a^2(t_2 - t_1)^2\,[a(t_2 + t_1)\cos\alpha + 2a\sin\alpha]^2 = c^2$

$a^2[(t_2 + t_1)^2 - 4t_1t_2]\,[a(t_2 + t_1)\cos\alpha + 2a\sin\alpha]^2 = c^2$]

Ex-3 Through each point of the straight line $x = my + h$ is drawn the chord of the parabola $y^2 = 4ax$ which is bisected at the point. Prove that it always touches parabola $(y + 2am)^2 = 8a(x - h)$.



[Sol. Equation of var. chord AB $y - y_i = \dfrac{2a}{y_i}(x - my_1 - h)$

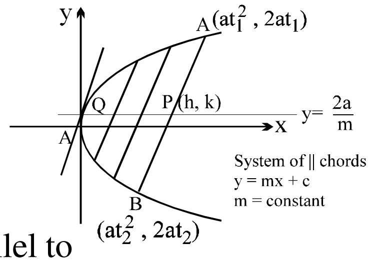
$yy_i - y_i^2 = 2ax - 2amy_1 - 2ah$

$y_i^2 - (y + 2am)y_i + 2a(x - h) = 0$

$(y + 2am)^2 = 8a(x - h)$]

DIAMETER : The locus of the middle points of a system of parallel chords of a

Parabola is called a ***Diameter***. Equation to the diameter of a parabola is y = 2a/m, where m = slope of parallel chords.



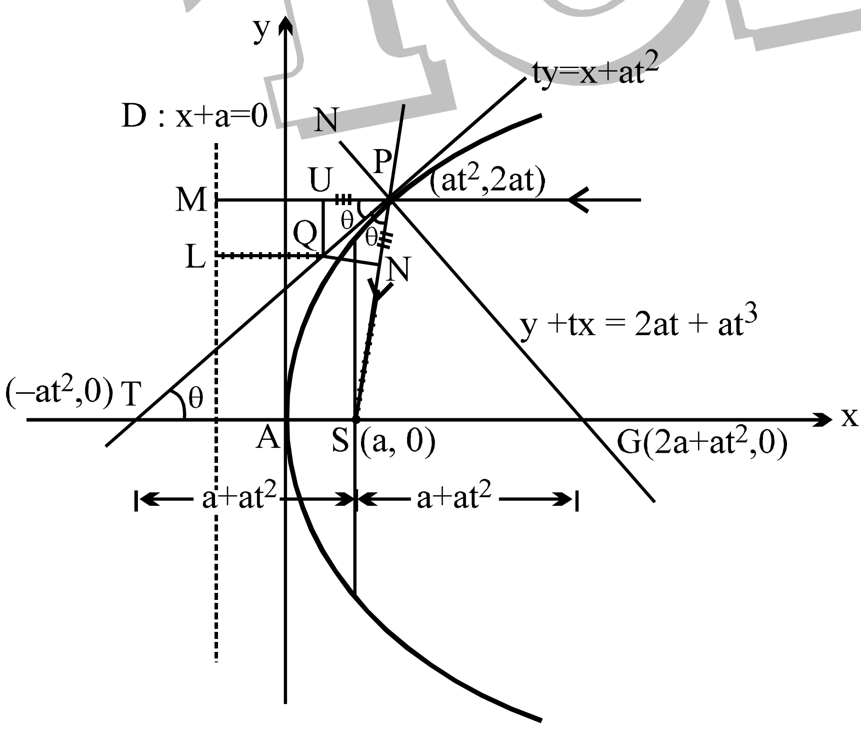
Explanation : Slope of AB is $m = \dfrac{2}{t_1 + t_2}$ ....(1)

also $2k = 2a\,(t_1 + t_2) \Rightarrow t_1 + t_2 = \dfrac{k}{a}\ \therefore k = \dfrac{2a}{m}$ = constant

Hence equation of the diameter is $y = \dfrac{2a}{m}$ i.e. a line parallel to the axis of the parabola. Solving $y = \dfrac{2a}{m}$ with $y^2 = 4ax$, we have, $\dfrac{4a^2}{m^2} = 4ax$ or $x = \dfrac{a}{m^2}$ Hence coordinates of Q are $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ Hence the tangent at the extremity of a diameter of a parabola is parallel to the system of chords it bisects. Since point of intersection of the two tangents are A and B is $at_1t_2,\ a(t_1 + t_2)$ or $\left(a\,t_1t_2, \dfrac{2a}{m}\right)$ Hence the tangent at the ends of any chords of a parabola meet on the diameter which bisects the chord. **Note: A line segment from a point P on the parabola and parallel to the system of parallel chords is called the ordinate to the diameter bisecting the system of parallel chords and the chords are called its double ordinate.**

IMPORTANT HIGHLIGHTS :

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then



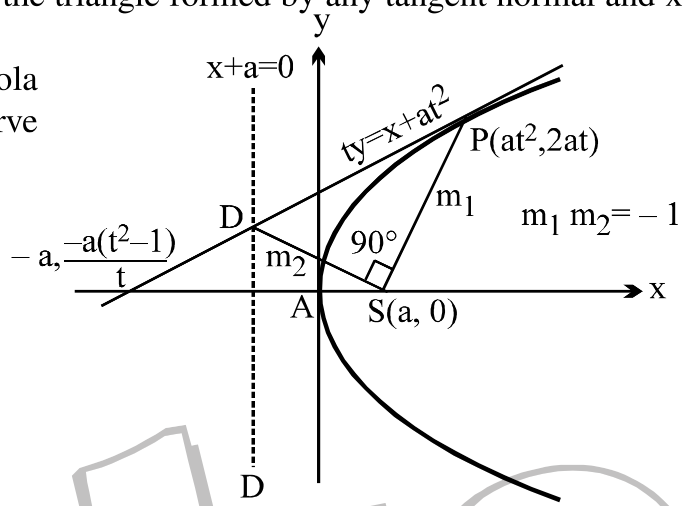
PM = PS
PU = PN
PM – PU = PS – PN
MU = SN
QL = SN

ST = SG = SP where ‘S’ is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & theperpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

Deduce that, if Q is any point on the tangent and QN is the perpendicular from Q on focal radius and QL is the perpendicular on the directrix then QL = SN.

☞ **Note :** Circle circumscribing the triangle formed by any tangent normal and x-axis, has its centre at focus.

(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.

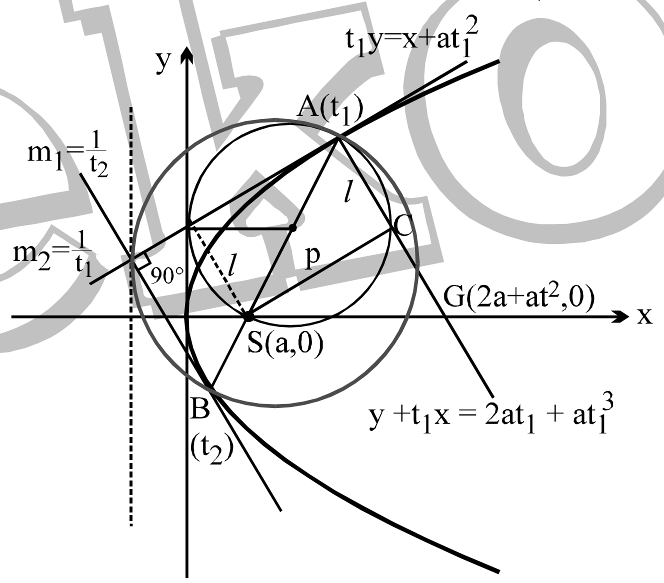


$$m_1 = \frac{2at}{at^2 - a} = \frac{2t}{t^2 - 1} \quad ;$$

$$m_2 = \frac{-a(t^2-1)}{t \cdot 2a} = -\frac{(t^2-1)}{2t}$$

$\therefore \qquad m_1 m_2 = -1 \qquad$]

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix**, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P (at^2, 2at) as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.



☞ **Note :** (1) For computing p draw a perpendicular from S (a, 0) on tangent at P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

i.e. locus of the feet of the perpendicular drawn from focus upon a variable tangent is the tangent drawn to the parabola at its vertex.

Explanation : Tangent 't' is

$ty = x + at^2$ .....(1)

Passing through (h, k) hence

$tk = h + at^2$ .....(A)

A line through (a, 0) with slope – t

$y = -t(x - a)$ ....(2)

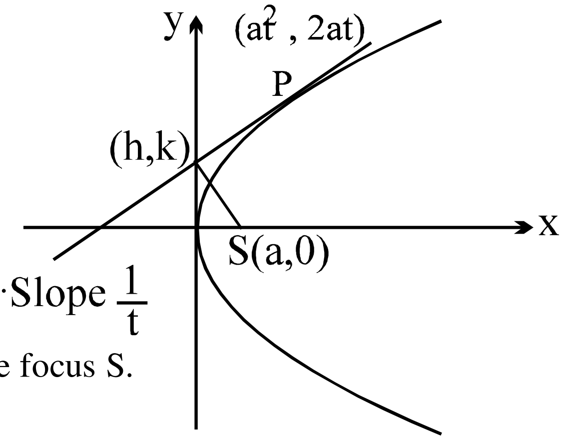
(2) also passes through (h, k)

$k = -th + at$

$tk = -t^2h + at^2$ ....(B)

(A) – (B) gives $0 = (1 + t^2)h$

$\Rightarrow$ x = 0 which is the tangent at the vertex.



(e) If the tangents at P and Q meet in T, then :

- TP and TQ subtend equal angles at the focus S.
- $ST^2 = SP \cdot SQ$ &
- The triangles SPT and STQ are similar.

(i) To prove that $\alpha = \beta$, it will be sufficient to prove that 'T' lies on the angle bisector of the angle $\angle PSQ$ i.e. perpendicular distance of 'T' from the line SP is equal to the perpendicular of T from SQ.

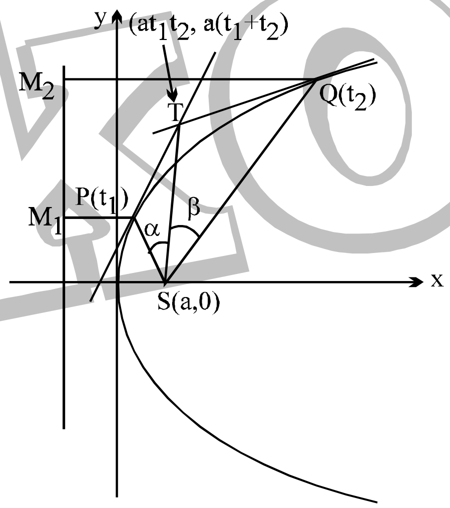
equation of SP

$$y = \frac{2at_1}{at_1^2 - a}(x - a)$$

$$2t_1x - (t_1^2 - 1)y - 2ab_1 = 0$$

$$p_1 = \left|\frac{2at_1^2t_2 - (t_1^2 - 1)a(t_1 + t_2) - 2at_1}{\sqrt{(t_1^2 - 1)^2 + 4t_1^2}}\right|$$

$$= a\,|\,t_1 - t_2\,|$$



|||ly $p_2 = a\,|\,t_2 - t_1\,| \Rightarrow \alpha = \beta$, Hence proved.

Now (ii) $SP \cdot SQ = (a + at_1^2)(a + at_2^2) = a^2(1 + t_1^2)(1 + t_2^2)$

also $(ST)^2 = a^2(t_1t_2 - 1)^2 + a^2(t_1 + t_2)^2 = a^2[t_1^2 t_2^2 + 1 + t_2^2 + t_2^2]$

$$= a^2(1 + t_1^2)(1 + t_2^2)$$

Hence $(ST)^2 = SP \cdot SQ$

This is conclusive that product of the focal radii of two points P and Q is equal to the square of the distance of focus from the point of intersection of the tangents drawn at P and Q.

(iii) again, $\frac{ST}{SP} = \frac{SQ}{ST}$ and $\alpha = \beta$

hence the two triangles SPT and SQT are similar.

(f) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being (– a, 0) & (3a, 0).

[Hint:

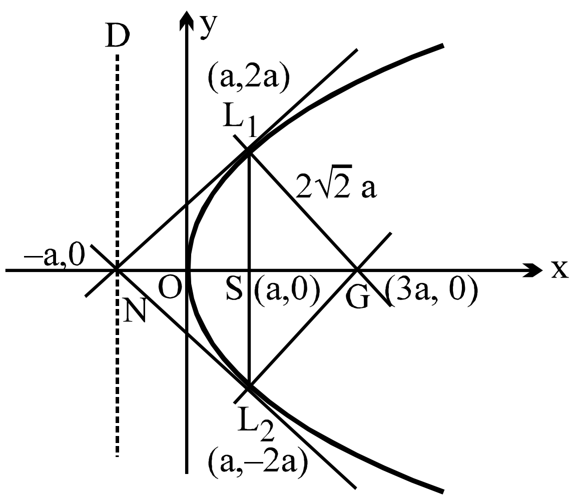


figure is Self explanatory]

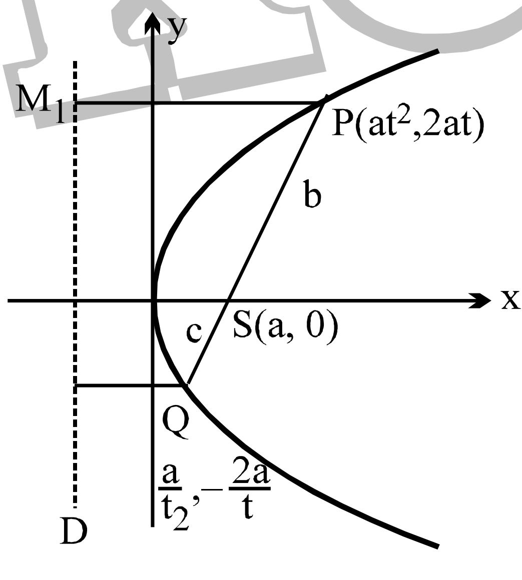
☞ **Note :**

(1) The two tangents at the extremities of focal chord meet on the foot of the directrix.

(2) Figure L_1NL_2G is square of side $2\sqrt{2}\,a$

(g) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is : $2a = \frac{2bc}{b+c}$ **i.e.** $\frac{1}{b}+\frac{1}{c}=\frac{1}{a}$.

[Sol. $2a = \frac{2bc}{b+c}$ i.e. $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$



$b = a + at^2$

$\Rightarrow$ $b = a(1 + t^2)$

$\Rightarrow$ $\frac{a}{b} = \frac{1}{1+t^2}$ ....(1)

$c = a + \frac{a}{t^2}$

$\Rightarrow$ $c = a\left(1+\frac{1}{t^2}\right) \Rightarrow \frac{a}{c} = \frac{t^2}{t^2+1}$ ....(2)

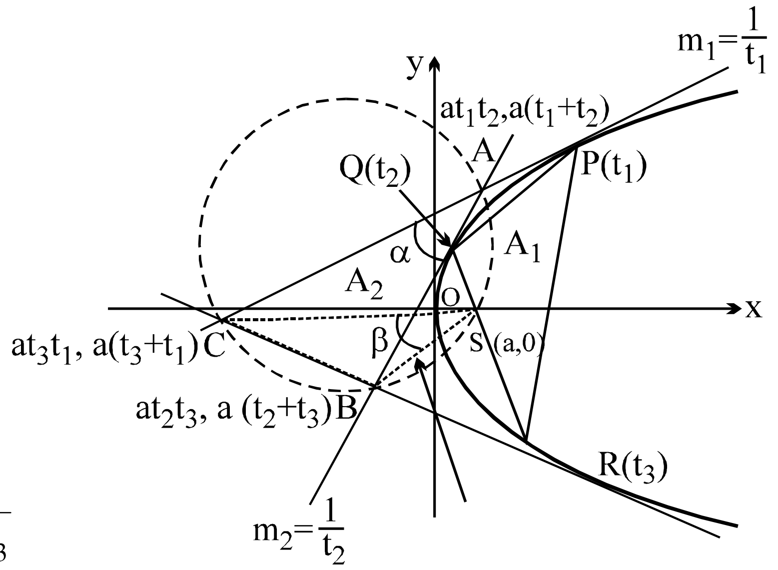
from (1) and (2)

$\frac{a}{b} + \frac{a}{c} = 1 \qquad \Rightarrow \qquad \frac{1}{a}=\frac{1}{b}+\frac{1}{c}$]

(h) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

TPT $\alpha = \beta$

$\frac{1}{t_1} = \tan\theta_1$; $\frac{1}{t_2} = \tan\theta_2$; $\frac{1}{t_3} = \tan\theta_3$



$\therefore \quad \tan\alpha = \left|\frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1 \cdot \tan\theta_2}\right|$

$\tan\alpha = |\tan(\theta_1 - \theta_2)|$ ....(1)

$m_1 = \frac{a(t_3+t_1)}{at_3t_1 - a} = \frac{t_3+t_1}{t_3t_1 - 1}$

$= \frac{\frac{1}{t_1}+\frac{1}{t_3}}{1-\frac{1}{t_1t_3}} = \frac{\tan\theta_1 + \tan\theta_3}{1-\tan\theta_1\cdot\tan\theta_3}$

$m_1 = \tan(\theta_1 + \theta_3)$

|||ly $\quad m_2 = \tan(\theta_2 + \theta_3)$

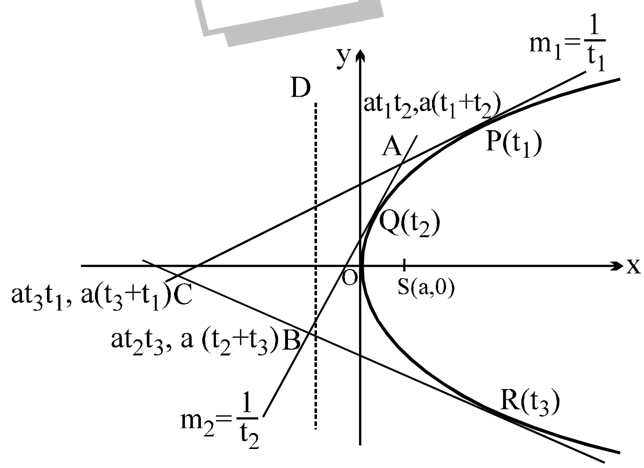
$\tan\beta = \left|\frac{\tan(\theta_1+\theta_3) - \tan(\theta_2+\theta_3)}{1+\tan(\theta_1+\theta_3)\cdot\tan(\theta_2+\theta_3)}\right|$

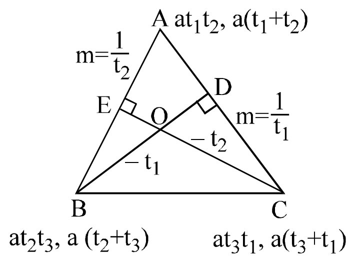
$\tan\beta = |\tan(\theta_1 - \theta_2)|$ ....(2)

from (1) and (2), we get

$\alpha = \beta$ hence proved]

(i) The orthocentre of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co–ordinates $-a,\ a(t_1 + t_2 + t_3 + t_1t_2t_3)$.



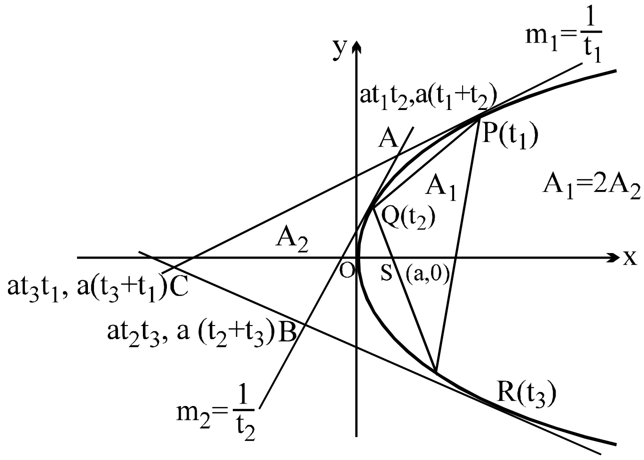


Find intersection of BD and CE to get 'O'.

(j) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Refer figure of point (h)

$$\frac{A_1}{A_2} = \frac{\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}}{\begin{vmatrix} at_1t_2 & a(t_1+t_2) & 1 \\ at_2t_3 & a(t_2+t_3) & 1 \\ at_3t_1 & a(t_3+t_1) & 1 \end{vmatrix}} = 2$$



MORE ABOUT NORMALS

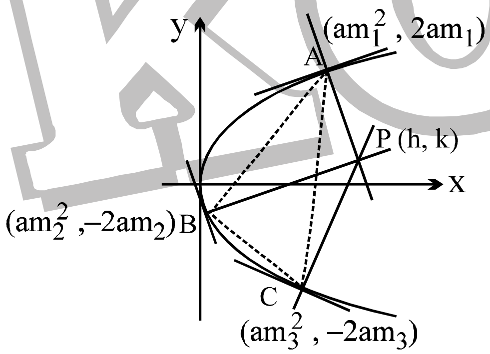
If normal drawn to a parabola passes through a point P(h , k) then
$k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$.

Then gives $m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; $m_1m_2m_3 = -\frac{k}{a}$.

where m_1, m_2, & m_3 are the slopes of the three concurrent normals. Note that the algebraic sum of the :

- slopes of the three concurrent normals is zero.
- ordinates of the three conormal points on the parabola is zero.
- Centroid of the triangle formed by three co−normal points lies on the x−axis.

Example : Consider of the triangle ABC is



$$x_1 = \frac{a(m_1^2 + m_2^2 + m_3^2)}{3}$$

$$y_1 = -\frac{2a(m_1 + m_2 + m_3)}{3} = 0$$

now $x_1 = \frac{a}{3}[(m_1 + m_2 + m_3)^2 - 2\sum m_1m_2]$

$$= -\frac{2a}{3}\frac{(2a-h)}{a} = \frac{2}{3}(h - 2a)$$

$\therefore$ centroid is $\frac{2}{3}(h - 2a), 0$ but $\frac{2}{3}(h - 2a) > 0$

$\therefore$ $h > 2a$

Hence abscissa of the point of concurrency of 3 concurrent normals > 2a.

EXAMPLES :

Ex.1 Find the locus of a point which is such that **(a)** two of the normals drawn from it to the parabola are at right angles, **(b)** the three normals through it cut the axis in points whose distances from the vertex are in arithmetical progression.

[Ans : (a) $y^2 = a(h - 3a)$; (b) $27ay^2 = 2(x - 2a)^3$] [Ex.237, Pg.212, Loney]

[Sol. (a) we have $m_1\, m_2 = -1$

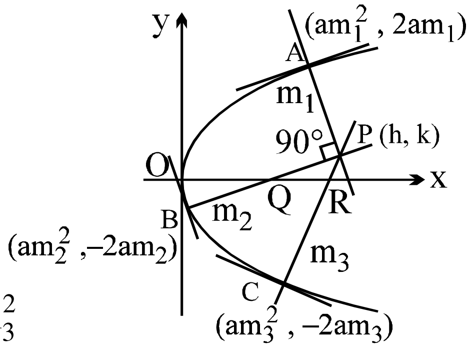
also $m_1\, m_2\, m_3 = -\frac{k}{a}$

$\therefore \quad m_3 = \frac{k}{a}$

put $m_3 = -\frac{k}{a}$ is a root of

$am^3 + (2a - h)m + k = 0$

(b) $\quad y = mx - 2am - am^3$



hence $\;2a + am_1^2,\ 2a + am_2^2,\ 2a + am_3^2$

$\therefore \quad 2m_2^2 = m_1^2 + m_3^2$

$$3m_2^2 = m_1^2 + m_2^2 + m_3^2 = (m_1 + m_2 + m_3)^2 - 2\left(\sum m_1m_2\right) = \frac{2(h-2a)}{a}$$

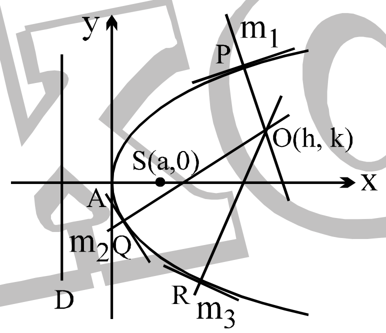
$$m_2^2 = \frac{2(h-2a)}{3a} \quad \text{which is root of } am^3 + (2a - h)m + k = 0 \qquad]$$

Ex.2 If the normals at three points P, Q and R meet in a point O and S be the focus, prove that SP · SQ · SR = a· SO². [Ex.238, Pg.213, Loney]

[Sol. $SP = a(1+m_1^2)$;

$SQ = a(1+m_2^2)$;

$SR = a(1+m_3^2)$



$$\frac{SP \cdot SQ \cdot SR}{a^3} = (1+m_1^2)(1+m_2^2)(1+m_3^2)$$

$$\left[1 + \left(\left(\sum m_1\right)^2 - 2\sum m_1m_2\right) + \left(\left(\sum m_1m_2\right)^2 - \underbrace{2m_1m_2m_3\left(\sum m_1\right)}_{\text{zero}}\right) + (m_1m_2m_3)^2\right]$$

Ex.3 A circle circumscribing the triangle formed by three co–normal points passes through the vertex of the parabola and its equation is, $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$. [Q.13, Ex-30, Loney]

[Sol. Equation of the normal at P

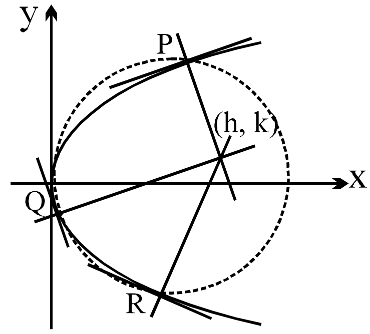
$y + tx = 2at + at^3$

passes through (h, k)

$at^3 + (2a - h)t - k = 0 \;\begin{cases} t_1 \\ t_2 \\ t_3 \end{cases} \qquad ....(1)$

$t_1 + t_2 + t_3 = 0$

$t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a-h}{a}, \quad t_1t_2t_3 = \frac{k}{a}$



Let the circle through PQR is

$x^2 + y^2 + 2gx + 2fy + c = 0$

solving circle $x = at^2$, $y = 2at$

$$a^2t^4 + 4a^2t^2 + 2gat^2 + 2f \cdot 2at + c = 0$$

$$a^2t^4 + 2a(2a + g)t^2 + 4fat + c = 0 \qquad ....(2)$$

$$t_1 + t_2 + t_3 + t_4 = 0$$

but $t_1 + t_2 + t_3 = 0 \quad \Rightarrow \quad t_4 = 0 \;\Rightarrow$ circle passes through the origin

hence the equation of the circle

$$x^2 + y^2 + 2gx + 2fy = 0$$

now equation (2) becomes

$$at^3 + 2(2a + g)t + 4f = 0 \quad (t_1, t_2, t_3) \qquad ....(3)$$

(1) and (3) must have the same root

$$2(2a + g) = 2a - h$$

$$2g = -(h + 2a)$$

and $4f = -k \quad \Rightarrow \quad 2f = -\dfrac{k}{2}$

Hence the equation of the circle is

$$x^2 + y^2 - (h + 2a)x - \frac{k}{2}y = 0 \quad \Rightarrow \quad 2(x^2 + y^2) - 2(h + 2a)x - ky = 0 \;]$$

Ex.4 Three normals are drawn to the parabola $y^2 = 4ax\cos\alpha$ from any point on the straight line $y = b\sin\alpha$. Prove that the locus of the orthocentre of the triangle formed by the corresponding tangent is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the angle α being variable.

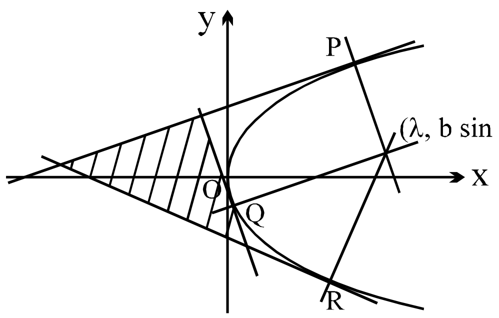
[Q.15, Ex.30, Loney]

[Sol. $y^2 = 4Ax$ where $A = a\cos\alpha$

$y + tx = 2At + at^3$ passes through $\lambda, b\sin\alpha$

$b\sin\alpha + t\lambda = 2At + At^3$

$At^3 + (2A - \lambda)t - b\sin\alpha = 0$



$\therefore \quad t_1 + t_2 + t_3 = 0 \quad ; \quad t_1 t_2 t_3 = \dfrac{b\sin\alpha}{A}$

also $h = -A = -a\cos\alpha \qquad ....(1)$

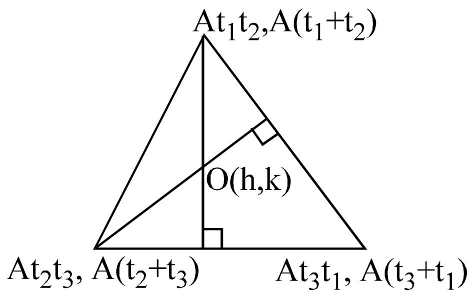
and $k = A(t_1 + t_2 + t_3 + t_1 t_2 t_3)$

$$= A\left(0 + \frac{b\sin\alpha}{A}\right)$$

$$k = b\sin\alpha \qquad ....(2)$$

from (1) and (2) locus is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;]$$



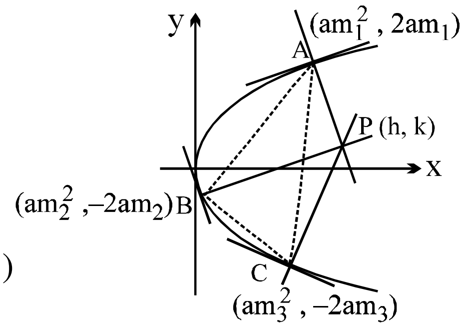
LEVEL 3 PROBLEMS

Ex.1 Locus of a point P when the 3 normals drawn from it are such that area of the triangle formed by their feet is constant. [Q.6, Ex-30, Loney]

[Hint: Area of Δ ABC = constant

$$\begin{vmatrix} am_1^2 & -2am_1 & 1 \\ am_2^2 & -2am_2 & 1 \\ am_3^2 & -2am_3 & 1 \end{vmatrix} = C$$

$\Rightarrow$ $(m_1 - m_2)^2 (m_2 - m_3)^2 (m_3 - m_1)^2 = C$ ....(1)



consider

$$(m_1 - m_2)^2 = m_1^2 - 2m_1m_2 + m_2^2$$

$$= -[m_1 (m_2 + m_3) + 2m_1m_2 + m_2(m_1 + m_3)$$

$$= -[(\sum m_1m_2) + 3m_1m_2] \quad (m_1m_2m_3 = -\frac{k}{a})$$

$$= \frac{h-2a}{a} + \frac{3k}{am_3}$$

$$\left[\underbrace{(h-2a)}_{x} + \underbrace{\frac{3k}{m_3}}_{l}\right]\left[(h-2a) + \underbrace{\frac{3k}{m_2}}_{m}\right]\left[(h-2a) + \underbrace{\frac{3k}{m_1}}_{n}\right]$$

hence equation (1) becomes

$(x + l)\ (x + m)\ (x + n) = \text{constant}$

$x^3 + (l + m + n)x^2 + (lm + mn + nl)x + lmn = 0$

$$(h-2a)^3 + 3k\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right)(h-2a)^2 + 9k^2\left(\frac{1}{m_1m_2} + \frac{1}{m_2m_3} + \frac{1}{m_3m_4}\right)(h-2a) + \frac{27k^3}{m_1m_2m_3}$$

= constant]

Ex.2 The sides of a triangle touch a parabola $y^2 = 4ax$ and two of its angular points lie on another parabola $y^2 = 4b(x + c)$ with its axis in the same direction, prove that the locus of the third angular point is another parabola. [Q.31, Ex-30, Loney]

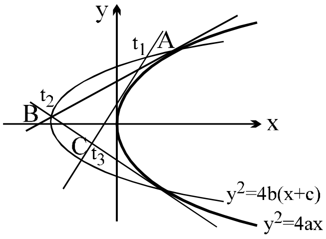
[Sol. consider the equations abtained by putting the coordinates of A and B in $y^2 = 4b(x + c)$

$a^2(t_1 + t_2)^2 = 4b(at_1t_2 + c)$ ....(1)

$a^2(t_2 + t_3)^2 = 4b(at_2t_3 + c)$ ....(2)

this implies that t_1 and t_3 are the roots of the equation,

$a^2(t + t_2)^2 = 4b(a\,t\,t_2 + c)$ having t_1 and t_3 as its roots



i.e. $a^2(t^2 + 2tt_2 + t_2^2) = 4batt_2 + 4bc$

$\Rightarrow$ $a^2t^2 + 2t(a^2t_2 - 2abt_2) + a^2 t_2^2 - 4bc = 0$

$\Rightarrow \quad t_1 + t_3 = -\dfrac{a^2t_2 - 2abt_2}{a^2} = \dfrac{t_2(2b-a)}{a}$

and $\quad t_1\, t_3 = \dfrac{a^2t_2^2 - 4bc}{a^2}$; $\quad$ hence $\dfrac{k}{a} = \dfrac{t_2(2b-a)}{a} \quad \Rightarrow \quad t_2 = \dfrac{k}{2b-a}$

$\dfrac{h}{a} = \dfrac{at_2^2 - 4bc}{a^2} \quad \Rightarrow \quad \dfrac{ah+4bc}{a} = t_2^2 \quad \Rightarrow \quad \dfrac{k^2}{(2b-a)^2} = \dfrac{ah+4bc}{a}$

hence locus is $\quad ay^2 = (2b - a)^2\,(ax + 4bc)$]

Ex.3 Circles are drawn through the vertex of the parabola to cut the parabola orthogonally at the other point of intersection. Prove that the locus of the centres of the circles is the curve,
$2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$ [Q.26, Ex.28 (Loney)]

Ex.4 If the normal at P and Q meet on the parabola, prove that the point of intersection of the tangents at P and Q lies either on a certain straight line, which is parallel to the tangent at the vertex, or on the curve whose equation is $y^2(x + 2a) + 4a^3 = 0$.
[Q.11, Ex.29 (Loney)]

Ex.5 (a) Prove that infinite number of triangles can be constructed in either of the parabolas $y^2 = 4ax$ and $x^2 = 4by$ whose sides touch the other parabola.

(b) Prove that the locus of the centre of the circle, which passes through the vertex of a parabola and through its intersections with a normal chord, is the parabola $2y^2 = ax - a^2$. [Q.25, Ex.30 (Loney)]

